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RESEARCH ARTICLE

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Dimension of certain sets of regular and minus continued fractions with positive partial quotients

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Abstract An estimate for the Hausdorff dimension of $x \in \mathbb{R}$ whose partial quotients of its regular continued fraction or minus continued fraction (MCF) are in $E \subseteq \mathbb{N}$ is given. This enables us to give a new proof for the Texan conjecture on $[0, \frac{1}{2}]$ which is valid for both regular and MCF. Also we show that if $E \subseteq \mathbb{N}$ and $\sum_{e \in E} \frac{1}{e} = \infty$, then the Hausdorff dimension of E is at least $\frac{1}{2}$.

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الملخص

يتم إعطاء تقريب لبعد هاوسدورف لـِ $x \in \mathbb{R}$ الذي تكون الكسور الجزئية لكسره المنتظم الممتد أو الممتد الناقص (MCF) في $E \subseteq \mathbb{N}$. يمكننا هذا من إعطاء إثبات جديد لحدس تكسان على $[0, \frac{1}{2}]$ والذي يتحقق لكل من الكسور االمنتظمة وَ MCF. نبين أيضاً أنه إذا كانت $E \subseteq \mathbb{N}$ وَ $\sum_{e \in E} \frac{1}{e} = \infty$، فإن بعد هاوسدورف لـِ E هو $\frac{1}{2}$ على الأقل.

1 Introduction

A real number $x \in \mathbb{R}$ can be represented in its regular continued fraction (RCF) form as

$$x = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{\ddots}}} \tag{1.1}$$

where $n_1 \in \mathbb{Z}$, $n_i \in \mathbb{N}$ for $i \geq 2$. Another form of continued fractions

$$x = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \cfrac{1}{\ddots}}} \tag{1.2}$$

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where $n_1 \in \mathbb{Z}$ and $n_i \in \mathbb{Z} \setminus \{0\}$ for $i \geq 2$, is called the minus continued fraction (MCF) or backward continued fraction. When $n_i \in \mathbb{Z}$, representing a real number with MCF is not unique [15,19]. However, it will be unique if $n_i \geq 2$ for $i \geq 2$ [15]. Due to this fact, in this paper all MCF's have positive partial quotient.

MCF like RCF has many applications, as an example see [8] for the utilization of MCF for the flows on the modular surface. Therefore, MCF appears as a limit set of the modular group.

Recall that a continued fraction of a real number can be obtained using the *Gauss map*. This map is usually accompanied by a measure called *Gauss measure*—the only measure preserved by the map—and also the Hausdorff dimension which is the main concern of this note. The Hausdorff dimension was first defined by Felix Hausdorff in 1918 and since then it has been used to understand the fractal dimensions of certain subsets of a set. In particular from the beginnings of introducing this dimension, it was used to understand the fractal dimensions of certain subsets of real numbers presented by continued fractions. Let for instance, $E_1^n = \{1, 2, \ldots, n\}$ and $C_{E_1^n} = \{x \in (0, 1) : x = \frac{1}{n_2+\frac{1}{\cdots}}, \ n_i \leq n, \ \forall i \geq 2\}$. In 1928, Jarnic [12] proved that for $n \geq 8$, $1 - \frac{4}{n \log 2} < \mathrm{Dim}_H C_{E_1^n} < 1 - \frac{1}{8n \log n}$ where $\mathrm{Dim}_H C_E$ stands for the Hausdorff dimension of E. In 1989, Hensley [10] showed that $\mathrm{Dim}_H C_{E_1^n} = 1 - \frac{6}{\pi^2 n} - \frac{72 \log n}{\pi^4 n^2} + O(\frac{1}{n^2})$. The reader may find some other recent results in [11], [13] and [14].

Bowen [1,2] uses the concept of pressure in thermodynamic formalism and gives a formula for computing the Hausdorff dimension (Theorem 3.1) which is used for our approach here. We consider a finite set $E \subset \mathbb{N}$ and give an estimate for the Hausdorff dimension of points represented by RCF or MCF whose partial quotients are in E. This is done in Theorem 3.3. Then in Theorem 3.7 and Remark 3.8 we extend the results to the case when $E_n^\infty = \{n, n+1, \ldots\}$ or $E = \{h(n), h(n+1), \ldots\}$ where h is an increasing positive integer valued function. In Theorem 3.10, we show that if $E \subseteq \mathbb{N}$ and $\sum_{e \in E} \frac{1}{e} = \infty$, then the Hausdorff dimension of E is at least $\frac{1}{2}$. By minor changes in the proof, the results are valid for both RCF and MCF.

We also give a new proof for the Texan conjecture on $[0, \frac{1}{2}]$ in Corollary 3.9. This conjecture states that $\{\mathrm{Dim}_H C_E : E \subset \mathbb{N}, |E| < \infty\}$ is dense in $[0, 1]$. For RCF, the conjecture was proved for $[0, \frac{1}{2}]$ in [17] and for $[0, 1]$ in [16]. Our proof is valid for both RCF and MCF.

2 Minus continued fractions

In this section, we introduce the Renyi map and the G-expansion as two algorithms for the generation of real numbers by MCF. In this paper, we only use the G-expansion which was first introduced in [8]; but before that a brief reminder of the older algorithm generated by the Renyi map is given.

2.1 The Renyi map and G-expansion

Let $R : [0, 1) \to [0, 1)$ be the Renyi map defined as

$$R(x) = \frac{1}{1-x} - \lfloor \frac{1}{1-x} \rfloor.$$

Each real number $x \in [0, 1)$ has a unique MCF expansion of the form

$$x = 1 - \cfrac{1}{n_1 + 1 - \cfrac{1}{n_2 + 1 - \cfrac{1}{\ddots}}}$$

where $n_i \in \mathbb{N}$. The partial quotient n_i is equal to $\lfloor \frac{1}{1-x_i} \rfloor$ where $x_1 = x$ and $x_i = R(x_{i-1})$. This algorithm is been more often used than the G-expansion which we are going to introduce.

The G-expansion of a real number is a unique MCF of the form (1.2) and will be denoted by

$$[n_1, n_2, n_3, \cdots] = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \cfrac{1}{\ddots}}} \tag{2.1}$$

where $n_1 \in \mathbb{Z}$ and $n_i \geq 2$ for $i \geq 2$. The partial quotients can be obtained by setting $x = x_1, n_i = \lceil x_i \rceil$ and $x_{i+1} = -\frac{1}{x_i - n_i}$, where $\lceil x_i \rceil$ is the smallest integer not less than x_i.

Conversely, it can be shown that given any infinite sequence of integers $n_1, n_2, \ldots$ with $n_i \geq 2$ for $i \geq 2$, $\lim_{k\to\infty} n_1 - \frac{1}{n_2 - \frac{1}{\cdots - \frac{1}{n_k}}}$ equals a real number x with G-expansion $[n_1,\ n_2,\ \ldots]$.

Note that if the G-expansion of $x \in (-1, 0]$ is $[0,\ n_1 + 1,\ n_2 + 1,\ \ldots]$, then the partial quotients of MCF for $x + 1 \in (0, 1]$ by the Renyi map are $1, n_1 + 1, n_2 + 1, \ldots$.

Let $\mathcal{A}$ be a set of countable letters and call any finite block of letters a word. The system (Σ, σ) is called one-sided (resp. two-sided) countable topological Bernoulli scheme (TBS) if $\Sigma = \mathcal{A}^{\mathbb{N}\cup\{0\}}$ (resp. $\Sigma = \mathcal{A}^{\mathbb{Z}}$). Here $\sigma : \Sigma \to \Sigma$ is defined by $\sigma\{a_i\} = \{a_{i+1}\}$ and is called the *shift map*.

Let $T(x) = x + 1$, $S(x) = -\frac{1}{x}$ and I_n^G be the interval $(n-1, n]$ for $n \geq 2$. Then $T^{n_i} S(I_{n_j}^G) \cap I_{n_i}^G \neq \emptyset$ for all $n_i, n_j \in \mathbb{N} \setminus \{1\}$ where $T^{n_i} = \overbrace{T \circ \cdots \circ T}^{n_i \text{ times}}$ and so $T^{n_i} S(I_{n_j}^G) = \{y \in \mathbb{R} : y = n_i - \frac{1}{x}, x \in I_{n_j}^G\}$. Since $T^{n_1} S(I_{n_2}^G) \cap I_{n_1}^G \neq \emptyset$, set $I_{n_1,n_2}^G := T^{n_1} S(I_{n_2}^G) \cap I_{n_1}^G$. Inductively, since $I_{n_2,\ldots,n_k}^G$ is defined and $T^{n_1} S(I_{n_2,\ldots,n_k}^G) \cap I_{n_1} \neq \emptyset$ then one can define the interval $I_{n_1,n_2,\ldots,n_k}^G$ as $T^{n_1} S(I_{n_2,\ldots,n_k}^G) \cap I_{n_1}$. The interval contains points with G-expansion having partial quotients $n_1, \ldots, n_k$ as their first k entries. Let Σ_G be the set of all one-sided sequences with entries coming from the G-expansions of irrationals in [0, 1).

2.2 Gauss map and RCF

Let $T : [0, 1] \to [0, 1]$ be the Gauss map defined as

$$T(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor,\ T(0) = 0.$$

Every irrational number has a unique regular continued faction whose partial quotients can be obtained by setting $x = x_1, n_i = \lfloor x_i \rfloor$ and $x_{i+1} = \frac{1}{x_i - n_i}$ [10]. Here $n_1 \in \mathbb{Z}$ and $n_i \in \mathbb{N}$ for $i \geq 2$. Note that for nonzero integers n_i we have

$$n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \frac{1}{\cdots}}}} = n_1 - \cfrac{1}{-n_2 - \cfrac{1}{n_3 - \cfrac{1}{-n_4 - \frac{1}{\cdots}}}}. \tag{2.2}$$

This allows us to use one notation for both RCF and MCF. Hence (1.1) will be written as $[n_1,\ -n_2,\ n_3,\ -n_4, \ldots]$. However, we are only dealing with positive integers.

3 Hausdorff dimension

Let X be a metric space. For each $s > 0$ and $\delta > 0$, define

$$\mathcal{H}_\delta^s = \inf\left\{\sum (\mathrm{diam} U_j)^s \mid U_j \text{ is a countable open } \delta - \text{cover of } X\right\}.$$

The limit $\mathcal{H}^s(X) = \lim_{\delta\to 0} \mathcal{H}_\delta^s(X)$ exists and may be infinite. The *Hausdorff dimension* of X is defined as

$$\mathrm{Dim}_H X := \inf\{s \geq 0 \mid \mathcal{H}^s(X) = 0\} = \sup\{s \geq 0 \mid \mathcal{H}^s(X) = \infty\}.$$

Let J be a finite interval and $f : J \to \mathbb{R}$ be a non-invertible $C^{1+\alpha}$ map where $\alpha \in (0, 1)$. To guarantee the existence of a Markov partition, we also assume that if $f(x) \in J$, then $|f'(x)| \geq r > 1$ for some fixed r.

Let $\varphi_i : J \to I_i, i = 1, \ldots, p$ be a finite number of the inverse branches of f where f is a Markov map on the intervals $\{I_i : i \in E\}$ and the intervals I_i are disjoint. Then $\varphi_1, \ldots, \varphi_p$ define an iterated function scheme (IFS) where $|\varphi(x) - \varphi(y)| \leq \frac{1}{r}|x - y|$ for all φ_i.

Recall that if f has the Markov property on the intervals I_i, then $f(I_i) \cap I_j \neq \emptyset$ implies that $I_j \subset f(I_i)$. So in this case, $f^{-1}(I_j) \subset I_i$ and $f^{-1}(I_j) \cap I_i = f^{-1}(I_j)$. If $\varphi_{n_i}(I_{n_j}) \cap I_{n_i} \neq \emptyset$ for some $i,\ j \in E$, set

$I_{n_i,n_j} := \varphi_{n_i}(I_{n_j}) \cap I_{n_i}$ to be an interval. Inductively, if $\varphi_{n_1}(I_{n_2,\ldots,n_k}) \cap I_{n_1} \neq \emptyset$, set $I_{n_1,\ldots,n_k} := \varphi_{n_1}(I_{n_2,\ldots,n_k}) \cap I_{n_1}$ to be an interval which is equal to $\varphi_{n_1} \ldots \varphi_{n_k}(J)$ by the Markov property of f. The limit set of the system,

$$\Lambda = \{x : x = \cap_{k=1}^{\infty} I_{n_1,\ldots,n_k} = \lim_{k\to\infty} \varphi_{n_1}\varphi_{n_2} \ldots \varphi_{n_k}(J)\}, \tag{3.1}$$

is an f-invariant Cantor set. By setting $\Sigma = \{(n_i)_{i=1}^{\infty} : \cap_{k=1}^{\infty} I_{n_1,\ldots,n_k} \neq \emptyset\}$, the map $\pi : \Sigma \to \Lambda$ is a bijection defined as $(n_i)_{i=1}^{\infty} \mapsto \lim_{k\to\infty} \varphi_{n_1}\varphi_{n_2} \ldots \varphi_{n_k}(J)$.

Let $\psi : \Sigma \to \mathbb{R}$ be a Hölder continuous function and σ the shift map on Σ. The *pressure* of ψ is defined as

$$\mathcal{P}(\psi) := \lim_{k\to\infty} \frac{1}{k} \log \sum_{C_{[n_1,\ldots,n_k]}} \exp \sup_{y\in C_{[n_1,\ldots,n_k]}} \sum_{j=0}^{k-1} \psi(\sigma^j y)$$

where the sum is over all cylinders $C_{[n_1,\ldots,n_k]} = \{y = (y_i)_{i=1}^{\infty} \in \Sigma | \, y_j = n_j, j = 1, \ldots, k\}$. It can be shown that for our case, the limit always exists. In our situation where f is expanding for the special family of potential functions $-s \log |f' \circ \pi|$, the pressure can be written as

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{C_{[n_1,\ldots,n_k]}} \exp\left(\sup_y \sum_{j=0}^{k-1} -s \log |f'(\pi\sigma^j y)|\right), \tag{3.2}$$

where the supremum is taken over $y \in C_{[n_1,\ldots,n_k]}$. One can use Ruelle–Perron–Frobenius Theorem to obtain Bowen's celebrated formula [1], [2]; see also [18] for different applications of this formula.

Theorem 3.1 (Bowen's Formula) *Let f be a $C^{1+\alpha}$ map, $\alpha \in (0, 1)$ and Λ the limit set of the system. Then $Dim_H \Lambda = s$ where s is the unique solution of Bowen's equation $\mathcal{P}(-s \log |f' \circ \pi|) = 0$.*

Remark 3.2 Consider (3.2) and let $x = \pi y$. Then by the Mean Value Theorem, $|(f^k)'(x)|^{-s} \asymp (\mathrm{diam} I_{n_1,\ldots,n_k})^s$, where $g(n) \asymp h(n)$ means both $\lim_{n\to\infty} \frac{g(n)}{f(n)}$ and $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ are bounded. So

$$\begin{aligned} P(s) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{C_{[n_1,\ldots,n_k]}} \exp\left(\sup_x \sum_{j=0}^{k-1} -s \log |f'(f^j x)|\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{C_{[n_1,\ldots,n_k]}} \exp\left(\sup_x -s \log \prod_{j=0}^{k-1} |f'(f^j x)|\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{C_{[n_1,\ldots,n_k]}} \sup_x |(f^k)'(x)|^{-s} \\ &= \lim_{n\to\infty} \frac{1}{n} \log \Sigma_{I_{n_1,\ldots,n_k}} (\mathrm{diam} I_{n_1,\ldots,n_k})^s. \end{aligned} \tag{3.3}$$

The formula (3.3) is quite handy when f' is constant. For instance consider the unit interval in base $m \geq 3$ with $f(x) = mx \bmod 1$. If E is a subset of $\{0, 1, \ldots, m-1\}$ then

$$\begin{aligned} P(s) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{n_i\in E} (\mathrm{diam} I_{n_1,\ldots,n_k})^s \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{n_i\in E} m^{-ns} \\ &= \lim_{n\to\infty} \frac{1}{n} \log(|E|^n m^{-ns}) \\ &= \log |E| - s \log m. \end{aligned}$$

Hence $s = \frac{\log |E|}{\log m}$.

We end this subsection by defining two sets:

$$C_E^R = \{x = [n_1, -n_2, n_3, \ldots] : n_i \in E(R) \subset \mathbb{N}\},$$
$$C_E^G = \{x = [n_1, n_2, n_3, \ldots] : n_i \in E(G) \subset \mathbb{N} \setminus \{1\}\},$$

where R and G stand for regular and G-expansion, respectively.

Estimates for Hausdorff dimensions of C_E^G and C_E^R

Let $E(G)$ and $E(R)$ be finite subsets of $\mathbb{N} \setminus \{1\}$ and $\mathbb{N}$, respectively. For $\alpha \in \{G, R\}$, let e_m and e_M be the smallest and largest elements of $E(\alpha)$, respectively.

In the next theorem, for the G-expansion case, that is when $\alpha = G$, by $[\overline{a}]$ we mean

$$a - \cfrac{1}{a - \frac{1}{\cdots}} = \frac{a + \sqrt{a^2 - 4}}{2}.$$

Also for $\alpha = R$, we need $a + \cfrac{1}{b + \frac{1}{a + \frac{1}{\cdots}}}$, $a,\ b \neq 0$ where by notations given in (2.2), it may be presented by

$$[\overline{a,\ -b}] = \frac{ab + \sqrt{(ab)^2 + 4ab}}{2b}.$$

Theorem 3.3 *Let $E(\alpha)$, e_m and e_M be as above. Then $t_m^{E(\alpha)} \leq Dim_H C_{E(\alpha)}^{\alpha} \leq t_M^{E(\alpha)}$ where $t_m = t_m^{E(\alpha)}$ and $t_M = t_M^{E(\alpha)}$ satisfy*

(i) $\sum_{e \in E(G)} \frac{1}{\left(e - \frac{1}{[\overline{e_M}]}\right)^{2t_m}} = 1$ *and* $\sum_{e \in E(G)} \frac{1}{\left(e - \frac{1}{[\overline{e_m}]}\right)^{2t_M}} = 1.$

(ii) $\sum_{e \in E(R)} \frac{1}{\left(e + \frac{1}{[\overline{e_m, -e_M}]}\right)^{2t_m}} = 1$ *and* $\sum_{e \in E(R)} \frac{1}{\left(e + \frac{1}{[\overline{e_M, -e_m}]}\right)^{2t_M}} = 1.$

Proof First we prove (i). Let $E := E(G)$, $I_{n_1, \ldots, n_k} := I^G_{n_1, \ldots, n_k}$, $t_m := t_m^{E(G)}$, $t_M := t_M^{E(G)}$ and $C_E := C_E^G$. Our intention is to use the formula for $P(s)$ given in (3.3). Hence the main task is to give an estimate for diam$I_{n_1, \ldots, n_k}$ which in turn leads to an estimate for Hausdorff dimension of C_E.

Recall $I_n = (n-1, n]$ and let $E = \{e_1, \ldots, e_\ell\}$. By choosing those I_{e_i} where $e_i \in E$, we have $C_E = \{x = \cap_{k=1}^{\infty} I_{n_1, \ldots, n_k} : n_i \in E\}$.

Denote by $(n_1, \ldots, n_k)$ the rational number $-\cfrac{1}{n_1 - \cfrac{1}{n_2 - \frac{1}{\ldots n_k}}}$. Then the diameter of $I_{n_1, \ldots, n_k}$ is $B_2^k - A_2^k$ where $A_2^k = (n_2, \ldots, n_k - 1)$ and $B_2^k = (n_2, \ldots, n_k)$. Also for a rational number $x = (a_1, \ldots, a_k)$ with $a_i \in E$ we have

$$-\frac{1}{[\overline{e_m}]} < -\underbrace{\cfrac{1}{e_m - \cfrac{1}{\cdots - \frac{1}{e_m}}}}_{k \text{ terms}} \leq -\cfrac{1}{a_1 - \cfrac{1}{\cdots - \frac{1}{a_k}}} \leq -\underbrace{\cfrac{1}{e_M - \cfrac{1}{\cdots - \frac{1}{e_M}}}}_{k \text{ terms}} < -\frac{1}{[\overline{e_M}]}. \tag{3.4}$$

We use a recursive argument to show that

$$\frac{\frac{1}{n_k(n_k-1)}}{\left(n_{k-1} - \frac{1}{[\overline{e_M}]}\right)^2 \left(n_{k-2} - \frac{1}{[\overline{e_M}]}\right)^2 \ldots \left(n_2 - \frac{1}{[\overline{e_M}]}\right)^2} < |I_{n_1, \ldots, n_k}| = B_2^k - A_2^k$$
$$< \frac{\frac{1}{n_k(n_k-1)}}{\left(n_{k-1} - \frac{1}{[\overline{e_m}]}\right)^2 \left(n_{k-2} - \frac{1}{[\overline{e_m}]}\right)^2 \ldots \left(n_2 - \frac{1}{[\overline{e_m}]}\right)^2}. \tag{3.5}$$

To show this first note that we have $A^k_{k-1} = (n_{k-1},\ n_k - 1)$, $B^k_{k-1} = (n_{k-1}, n_k)$ and so $B^k_{k-1} - A^k_{k-1} = \frac{1}{n_k(n_k-1)}$. Also $B^k_{k-2} - A^k_{k-2} = \frac{B^k_{k-1}-A^k_{k-1}}{(n_{k-1}-\frac{1}{n_k})(n_{k-1}-\frac{1}{n_k-1})}$. Applying (3.4) we have $(n_{k-1} - \frac{1}{[e_m]})^2 < (n_{k-1} - \frac{1}{n_k})(n_{k-1} - \frac{1}{n_k-1}) < (n_{k-1} - \frac{1}{[e_M]})^2$ and hence

$$\frac{\frac{1}{n_k(n_k-1)}}{\left(n_{k-1} - \frac{1}{[e_M]}\right)^2} < B^k_{k-2} - A^k_{k-2} < \frac{\frac{1}{n_k(n_k-1)}}{\left(n_{k-1} - \frac{1}{[e_m]}\right)^2}. \tag{3.6}$$

Now $B^k_{k-3} - A^k_{k-3} = \frac{B^k_{k-2}-A^k_{k-2}}{\left(n_{k-2}-\frac{1}{n_{k-1}-\frac{1}{n_k}}\right)\left(n_{k-2}-\frac{1}{n_{k-1}-\frac{1}{n_k-1}}\right)}$. Again applying (3.4) and (3.6) we will have an estimate for $B^k_{k-3} - A^k_{k-3}$ and repeating this procedure (3.5) will be established.

By the same argument as in the end of Sect. 2.1, we have a TBS system. This means n_j in $I_{n_1,\dots,n_j,\dots,n_k}$ can be any element of E. Therefore, raising (3.5) to the tth power, summing over all $n_1, \dots, n_k$, and taking the limit gives

$$\log\left(\sum_{e\in E} \frac{1}{(e - \frac{1}{[e_M]})^{2t}}\right) \le \lim_{k\to\infty} \frac{1}{k} \log \sum_{n_i\in E} |I_{n_1,\dots,n_k}|^t \le \log\left(\sum_{e\in E} \frac{1}{(e - \frac{1}{[e_m]})^{2t}}\right). \tag{3.7}$$

This establishes (i).

For the proof of (ii), we use the notation $[n_1,\ -n_2,\ n_3,\ \dots]$ for (1.1). By considering the following estimate instead of (3.4)

$$\frac{1}{[e_M,\ -e_m]} < \frac{1}{n_1 + \frac{1}{\dots+\frac{1}{n_k}}} < \frac{1}{[e_m,\ -e_M]},$$

and doing the same procedure as in case (i), we will have our conclusion.

If $2 \in E$ and $|E| \ge 2$, the right hand side of inequality (3.7) never vanishes for $t \in [0,\ 1]$. Hence t_M cannot be estimated and in this case we let $t_M = 1$. Note that in any other case $t_M < 1$.

It is convenient to define

$$b^E_m(t) = \sum_{e\in E} \frac{1}{\left(e - \frac{1}{[e_M]}\right)^{2t}}; \quad b^E_M(t) = \sum_{e\in E} \frac{1}{\left(e - \frac{1}{[e_m]}\right)^{2t}}$$

for $E \subset \mathbb{N}$, $t \ge 0$. □

Remark 3.4 Let E and E' be subsets of $\mathbb{N}$ and let t_m, t_M , t'_m and t'_M be the corresponding bounds for the Hausdorff dimensions of C^G_E and $C^G_{E'}$. Suppose $E := \{e_1,\ \dots,\ e_\ell\}$ and $E' := \{e'_1,\ \dots,\ e'_\ell\}$ with $e_i < e'_i$. Then $\mathrm{Dim}_H C^G_{E'} < \mathrm{Dim}_H C^G_E$. In fact, we have $b^{E'}_M(t) < b^E_m(t)$. This is because, e_i and e'_i are natural numbers and so $\frac{1}{e'_i - \frac{1}{[e'_m]}} < \frac{1}{e_i - \frac{1}{[e_M]}}$. Hence, $\sum_{e'_i\in E'} \frac{1}{\left(e'_i - \frac{1}{[e'_m]}\right)^{2t}} < \sum_{e_i\in E} \frac{1}{\left(e_i - \frac{1}{[e_M]}\right)^{2t}}$ which yields $b^{E'}_M(t) < b^E_m(t)$.

The following corollary gives a rougher estimate which is fairly suitable when E consists of consecutive integers $\{e_m, e_m + 1, \dots, e_M\}$ and e_m is large.

Corollary 3.5 *Let t'_m and t'_M be the solutions of $\beta(t) = \frac{(e_M)^{-2t+1}-(e_m)^{-2t+1}}{-2t+1} = 1$ and $\alpha(t) = \frac{(e_M)^{-2t+1}-(e_m-2)^{-2t+1}}{-2t+1} = 1$, respectively. Then $t'_m < \mathrm{Dim}_H C^G_E < t'_M$.*

Proof Using the inequality in (3.7) we have

$$
\begin{aligned}
\lim_{k\to\infty}\frac{1}{k}\log\sum_{n_i\in E}|I_{n_1,\ldots,n_{k-1}}|^t &\geq \log\left(\sum_{e\in E}\frac{1}{\left(e-\frac{1}{[e_M]}\right)^{2t}}\right)\\
&> \log\left(\int_{e_m}^{e_M+1}\frac{\mathrm{d}x}{\left(x-\frac{1}{[e_M]}\right)^{2t}}\right)\\
&= \log\left(\frac{\left(e_M+1-\frac{1}{[e_M]}\right)^{-2t+1}-\left(e_m-\frac{1}{[e_M]}\right)^{-2t+1}}{-2t+1}\right)\\
&> \log\left(\frac{(e_M)^{-2t+1}-(e_m)^{-2t+1}}{-2t+1}\right)=\log\beta(t).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\lim_{k\to\infty}\frac{1}{k}\log\sum_{n_i\in E}|I_{n_1,\ldots,n_{k-1}}|^t &\leq \log\left(\sum_{e\in E}\frac{1}{\left(e-\frac{1}{[e_m]}\right)^{2t}}\right)\\
&< \log\left(\int_{e_m-1}^{e_M}\frac{\mathrm{d}x}{\left(x-\frac{1}{[e_m]}\right)^{2t}}\right)\\
&= \log\left(\frac{\left(e_M-\frac{1}{[e_m]}\right)^{-2t+1}-\left(e_m-1-\frac{1}{[e_m]}\right)^{-2t+1}}{-2t+1}\right)\\
&< \log\left(\frac{(e_M)^{-2t+1}-(e_m-2)^{-2t+1}}{-2t+1}\right)=\log\alpha(t).
\end{aligned}
$$

Remark 3.6 Let t'_m and t'_M be the solution of $\beta(t)=\frac{(e_M)^{-2t+1}-(e_m)^{-2t+1}}{-2t+1}=1$ and $\alpha(t)=\frac{(e_M)^{-2t+1}-(e_m-2)^{-2t+1}}{-2t+1}=1$, respectively. Then $t'_m<\mathrm{Dim}_H C_E^R<t'_M$.

As mentioned in Sect. 1, Jarnic [12] gave an estimate for $\mathrm{Dim}_H C_{E_1^n}$ and then Hensley [10] improved that estimate. In the next result, we consider $E_n^\infty=\{n,\ n+1,\ \ldots\}$ and will give an estimate for $\mathrm{Dim}_H C_{E_n^\infty}$ for RCF and MCF. We will use the Lambert W function, also called the Omega function, defined to be the inverse function of $f(w)=we^w$. This is a single valued function for non-negative real numbers with derivative $W'(x)=\frac{W(x)}{x(1+W(x))}$ [4]. See also [9] for some applications.

Theorem 3.7 *Let $\alpha=G,R$ and $E_n^\infty=\{n,n+1,\ldots\}$ where $n\in\mathbb{N}\setminus\{1,\ 2\}$ when $\alpha=G$ and $n\in\mathbb{N}\setminus\{1\}$ for $\alpha=R$. Then*

$$
t_m^{E_n^\infty}=\frac{W(\ln n)+\ln n}{2\ln n};\ t_M^{E_n^\infty}=\frac{W(\ln(n-1+\epsilon(\alpha)))+\ln(n-1+\epsilon(\alpha))}{2\ln(n-1+\epsilon(\alpha))}
$$

where $W(x)$ is the Lambert W function and $\epsilon(G)=-\frac{1}{[n]}$ and $\epsilon(R)=-\frac{1}{n}$. Here $t_M^{E_n^\infty}-t_m^{E_n^\infty}=o(\frac{1}{n(\ln n)^a})$ for any $0<a<2$.

Proof We prove the theorem for $\alpha=G$. The proof for the other case is similar. Let $E_n^\infty=\cup_{\ell>n}E_n^\ell$ where $E_n^\ell=\{n,\ n+1,\ \ldots,\ \ell\}$. Then by the definition of Hausdorff dimension, $\mathrm{Dim}_H C^G_{E_n^\infty}=\sup_\ell \mathrm{Dim}_H C^G_{E_n^\ell}$. Also we have $E_n^\ell\subset E_n^{\ell+1}$ which implies $C^G_{E_n^\ell}\subset C^G_{E_n^{\ell+1}}$ and so $\mathrm{Dim}_H C^G_{E_n^\ell}$ is increasing with respect to ℓ. Therefore,

$\mathrm{Dim}_H C^G_{E_n^\infty} = \lim_{\ell\to\infty} \mathrm{Dim}_H C^G_{E_n^\ell}$. Let t_m^ℓ and t_M^ℓ be the corresponding bounds for $\mathrm{Dim}_H C^G_{E_n^\ell}$ obtained in Corollary 3.5. Both t_m^ℓ and t_M^ℓ are increasing and so $\lim_{\ell\to\infty} t_m^\ell \le \mathrm{Dim}_H C^G_{E_n^\infty} \le \lim_{\ell\to\infty} t_M^\ell$. Assume $t > \frac{1}{2}$, then $t_m^{E_n^\infty}$ is the solution of $\lim_{\ell\to\infty}\int_n^{\ell+1} \frac{\mathrm{d}x}{(x-\frac{1}{[\ell]})^{2t}} = \lim_{\ell\to\infty} \frac{(\ell+1)^{-2t+1}-n^{-2t+1}}{-2t+1} = 1$. This means $t_m^{E_n^\infty}$ is the solution for $\frac{-n^{-2t+1}}{-2t+1} = 1$, that is, $t_m^{E_n^\infty} = \frac{W(\ln n)+\ln n}{2\ln n}$.

On the other hand, the solution for $\lim_{\ell\to\infty}\int_{n-1}^{\ell} \frac{\mathrm{d}x}{(x-\frac{1}{[n]})^{2t}} = \frac{-(n-1-\frac{1}{[n]})^{-2t+1}}{-2t+1} = 1$ gives $t_M^{E_n^\infty} = \frac{W(\ln(n-1-\frac{1}{[n]}))+\ln(n-1-\frac{1}{[n]})}{2\ln(n-1-\frac{1}{[n]})}$.

To end the proof, note that $t_M^{E_n^\infty} - t_m^{E_n^\infty} = -(\frac{1}{2}+\frac{1}{2[n]})(\frac{W(\xi)}{\ln(\xi)})' = h(\xi)$ for some $\xi \in [n-1-\frac{1}{[n]}, n]$. But then $\lim_{\xi\to\infty} \xi(\ln(\xi))^a h(\xi) = 0$ for any $0 < a < 2$. □

Remark 3.8 The same procedure as in the proof of Theorem 3.7 may be used to obtain estimates for Hausdorff dimensions of some other infinite subsets of $\mathbb{N}$. Here we present one which will be used in the next corollary. Let $E_n^\infty(\frac{p}{q}) = \{\lfloor n^{\frac{p}{q}}\rfloor, \lfloor (n+1)^{\frac{p}{q}}\rfloor, \ldots\}$ where $p, q \in \mathbb{N}$, $p > q$. Fix $\frac{p}{q}$ and let $t_m^\infty = t_m^\infty(\frac{p}{q})$ and $t_M^\infty = t_M^\infty(\frac{p}{q})$ be the corresponding estimates for lower and upper bounds of $\mathrm{Dim}_H C^G_{E_n^\infty(\frac{p}{q})}$. We have

$$I_m(t) = \int_n^\infty \frac{\mathrm{d}x}{(x^{\frac{p}{q}})^{2t}} < \sum_{i=n}^\infty \frac{1}{(i^{\frac{p}{q}})^{2t}} \le \sum_{i=n}^\infty \frac{1}{\lfloor i^{\frac{p}{q}}\rfloor^{2t}} \le b_m^{E_n^\infty(\frac{p}{q})}(t).$$

Suppose $t > \frac{q}{2p}$, then $I_m(t) = -\frac{n^{-2t\frac{p}{q}+1}}{-2t\frac{p}{q}+1}$ and let t'^∞_m be the solution of $I_m(t) = 1$. Then $t'^\infty_m = \frac{q}{2p}\left(\frac{W(\ln n)}{\ln n}+1\right)$ which must be less than t_m^∞, that is, the solution of $b_m^{E_n^\infty(\frac{p}{q})}(t) = 1$. In particular, for any $p, q, n \in \mathbb{N}$, $p > q$, $\mathrm{Dim}_H C^G_{E_n^\infty(\frac{p}{q})} > \frac{q}{2p}$.

On the other hand, $\left(\lfloor i^{\frac{p}{q}}\rfloor - \frac{1}{\lfloor n^{\frac{p}{q}}\rfloor}\right) > (\lfloor i^{\frac{p}{q}}\rfloor - 1) \ge (i-1)^{\frac{p}{q}}$. So

$$b_M^{E_n^\infty(\frac{p}{q})}(t) = \sum_{i=n}^\infty \frac{1}{\left(\lfloor i^{\frac{p}{q}}\rfloor - \frac{1}{\lfloor n^{\frac{p}{q}}\rfloor}\right)^{2t}} < \sum_{i=n}^\infty \frac{1}{((i-1)^{\frac{p}{q}})^{2t}} \le \int_n^\infty \frac{\mathrm{d}x}{((x-2)^{\frac{p}{q}})^{2t}} = I_M(t).$$

Let t'^∞_M be the solution of $I_M(t) = 1$. Then $t'^\infty_M = \frac{q}{2p}\left(\frac{W(\ln(n-2))}{\ln(n-2)}+1\right) > t_M^\infty$.

The Texan conjecture states that $\{\mathrm{Dim}_H C_E : E \subset \mathbb{N}\setminus\{1\}, |E| < \infty\}$ is dense on $[0, 1]$. This conjecture was first proved for $[0, \frac{1}{2}]$ by Mauldin and Urbanski [17] and then completed by Kessebohmer and Zhu [16] and Ghenciu [6]. Their proofs were for the RCF. The following corollary gives a new proof which shows that the conjecture is true for MCF on $[0, \frac{1}{2}]$. This proof also works for RCF on $[0, \frac{1}{2}]$.

Corollary 3.9 *The set* $\{\mathrm{Dim}_H C^G_E : E \subset \mathbb{N}\setminus\{1, 2\}, |E| < \infty\}$ *is dense in* $[0, \frac{1}{2}]$.

Proof We adopt the notations used in the above Remark and for each $n \in \mathbb{N}$, we define $E_n^\ell(\frac{p}{q}) = \{\lfloor n^{\frac{p}{q}}\rfloor, \lfloor (n+1)^{\frac{p}{q}}\rfloor, \ldots, \lfloor (n+\ell)^{\frac{p}{q}}\rfloor\}$. Let t_m^ℓ and t_M^ℓ be the corresponding bounds for $\mathrm{Dim}_H C^G_{E_n^\ell(\frac{p}{q})}$. To prove the corollary, we will show that $\{\mathrm{Dim}_H C^G_{E_n^\ell(\frac{p}{q})} : \ell, n, p, q \in \mathbb{N}, p > q\}$ is dense in $[0, \frac{1}{2}]$. Note that we have $\lim_{n\to\infty} \frac{W(\ln n)}{\ln n} \to 0$. So, we can choose n large enough such that $\frac{qW(\ln(n-2))}{2p\ln(n-2)} \le \epsilon$. Moreover, since $t_m^\ell \to t_m^\infty$, we can choose $\ell > n$ large enough such that $t_m^\infty - \epsilon \le t_m^\ell$. Then,

$$\frac{q}{2p} - \epsilon \le t'^\infty_m - \epsilon < t_m^\infty - \epsilon \le t_m^\ell \le \mathrm{Dim}_H C^G_{E_n^\ell(\frac{p}{q})} \le t_M^\ell \le t_M^\infty < t'^\infty_M \le \frac{q}{2p} + \epsilon.$$

This together with the fact that $\{\frac{q}{2p} : p, q \in \mathbb{N}, p > q\}$ concludes the proof. □

Let $\mathcal{F} = \{D \subset \mathbb{N} : \sum_{d \in D} \frac{1}{d} = \infty\}$. This is a family of subsets of $\mathbb{N}$. (A set of subsets $\mathcal{F}$ of $\mathbb{N}$ is a family if $A \subseteq B$ and $A \in \mathcal{F}$ then $B \in \mathcal{F}$.) This family is famous for Erdös conjectured that if $D \in \mathcal{F}$ then D has arbitrary long arithmetic progression [7]. If P is the set of prime numbers then $P \in \mathcal{F}$ and Erdös conjecture has been proved positively for P [5]. From qualitative combinatorial number theory, elements of $\mathcal{F}$ can be considered large. We prove that in our case they also have large Hausdorff dimension.

Theorem 3.10 *Suppose $D \subseteq \mathbb{N}$ and $\sum_{d \in D} \frac{1}{d} = \infty$. Then $Dim_H C_D^\alpha \geq \frac{1}{2}$ where $\alpha = G, R$.*

Proof First note the following facts that are immediate from the basic formula for Hausdorff dimension given in (3.3).

Let E be any subset of $\mathbb{N}$ and $|E| < \infty$ then

(i) Moving an element of E to right will decrease the Hausdorff dimension. By this we mean that if $e \in E$ and $E' = E \setminus \{e\} \cup \{e + k\}$ for some $k \in \mathbb{N}$, then $\mathrm{Dim}_H C_{E'}^\alpha \leq \mathrm{Dim}_H C_E^\alpha$.

(ii) If $E' \subset E$, $\mathrm{Dim}_H C_{E'}^\alpha \leq \mathrm{Dim}_H C_E^\alpha$.

To prove the theorem let $E_1^\infty(\beta) = \{1, \lfloor 2^\beta \rfloor, \ldots, \lfloor k^\beta \rfloor, \ldots\}$, $\beta > 1$. Since $\sum_{e \in E_1^\infty(\beta)} \frac{1}{e} < \infty$ we have

$$\lim_{n \to \infty} \frac{|D \cap \{1, \ldots, n\}|}{|E_1^\infty(\beta) \cap \{1, \ldots, n\}|} = \infty.$$

Hence fix $N \in \mathbb{N}$ such that for $n > N$,

$$|D \cap \{1, 2, \ldots, n\}| > 2|E_1^\infty(\beta) \cap \{1, 2, \ldots, n\}|. \tag{3.8}$$

Let $\lfloor k^\beta \rfloor$ be the least integer in $E_1^\infty(\beta)$ such that $\lfloor k^\beta \rfloor > N$ and set $E_k^\infty(\beta) := E_1^\infty(\beta) \setminus \{1, \ldots, \lfloor (k-1)^\beta \rfloor\} = \{\lfloor k^\beta \rfloor, \lfloor (k+1)^\beta \rfloor, \ldots\}$, $D_i := D \cap \{1, \ldots, \lfloor (k+i)^\beta \rfloor\}$, $i = 0, 1, 2, \ldots$. Now choose an element $d_0 \in D_0$ and replace D_0 with $D_0' = (D_0 \setminus \{d_0\}) \cup \{\lfloor k^\beta \rfloor\}$; that is as if we move d_0 to $\lfloor k^\beta \rfloor$th place. Also choose $d_1 \in D_1 \setminus \{d_0, \lfloor k^\beta \rfloor\}$ and let $D_1' = (D_1 \setminus \{d_0, d_1\}) \cup \{\lfloor k^\beta \rfloor, \lfloor (k+1)^\beta \rfloor\}$. This is possible because of (3.8). By induction and applying (3.8), let $d_n \in D_n \setminus \{d_0, \ldots, d_{n-1}, \lfloor k^\beta \rfloor, \ldots, \lfloor (k+n-1)^\beta \rfloor\}$ and let

$$D_n' = (D_n \setminus \{d_0, d_1, \ldots, d_n\}) \cup \{\lfloor k^\beta \rfloor, \lfloor (k+1)^\beta \rfloor, \ldots, \lfloor (k+n)^\beta \rfloor\}, \quad D' = \cup_{n=0}^\infty D_n'.$$

Hence $\{\lfloor k^\beta \rfloor, \lfloor (k+1)^\beta \rfloor, \ldots\} \subseteq D'$ and D' is constructed from D by moving some elements to right. By applying (i) and (ii) we have

$$\mathrm{Dim}_H C_{E_k^\infty(\beta)}^\alpha \leq \mathrm{Dim}_H C_{D'}^\alpha \leq \mathrm{Dim}_H C_D^\alpha.$$

Now we claim that $\mathrm{Dim}_H C_{E_k^\infty(\beta)}^\alpha > \frac{1}{2\beta}$. Then the conclusion follows from the fact that β can be arbitrary close to one.

The proof of the claim has the same reasoning similar to the argument in Remark 3.8. We have

$$\int_k^\infty \frac{\mathrm{d}x}{(\lfloor x^\beta \rfloor)^{2t}} > \int_k^\infty \frac{\mathrm{d}x}{(x^\beta)^{2t}} = -\frac{k^{-2\beta t+1}}{-2\beta t + 1}.$$

Thus a lower bound for Hausdorff dimension is a t_m where $k^{-2\beta t_m + 1} = 2\beta t_m - 1$, that is,

$$t_m = \frac{W(\ln k)}{2\beta \ln k} + \frac{1}{2\beta} > \frac{1}{2\beta}. \tag{3.9}$$

□

Also a class of examples satisfying the above theorem is when

$$\overline{d}(D) = \limsup \frac{|D \cap \{1, \dots, n\}|}{n} > 0,$$

because then $\sum_{d \in D} \frac{1}{d} = \infty$ [3].

Note, however, that there is not any positive lower bound for arbitrary infinite subsets of $\mathbb{N}$. For instance, if $m > 3$ and $E_m = \{2^n : n \in \mathbb{N}, n > m - 1\}$, then the same routine as in the Remark 3.8 yields

$$\int_{m-1}^{\infty} \frac{\mathrm{d}x}{\left(2^x - \frac{1}{[2^m]}\right)^{2t}} < \int_{m}^{\infty} \frac{\mathrm{d}x}{(2^x)^{2t}} = \frac{(2^m)^{-2t}}{t \ln 4} = \alpha(t).$$

From $\alpha(t) = 1$ we will have

$$t_M^{E_m} < \frac{W(m)}{2m \ln 2}$$

which tends to zero as m gets large.

Also a direct computation shows that if $E = \{2, 3, 4\}$, then t_m the lower bound for Hausdorff dimension, is larger than 0.613. Hence there are subsets of $\mathbb{N}$ with $\sum_{e \in E} \frac{1}{e} < \infty$ but having the Hausdorff dimension larger than $\frac{1}{2}$.

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